An alternate characterization of reflections

Suppose the line l makes an angle ψ with the positive x-axis, then the reflection about l can be written as

$$S_l = S_\psi = \begin{pmatrix} \cos 2\psi & \sin 2\psi \\ \sin 2\psi & -\cos 2\psi \end{pmatrix}$$

$$O(2) = \{ R_\theta S_\psi \mid 0 \le \theta < 2\pi,\ 0 \le \psi < \pi \}$$

The physical symmetry group of a circle is only rotations $SO(2) = \{ R_\theta \mid 0 \le \theta < 2\pi \}$

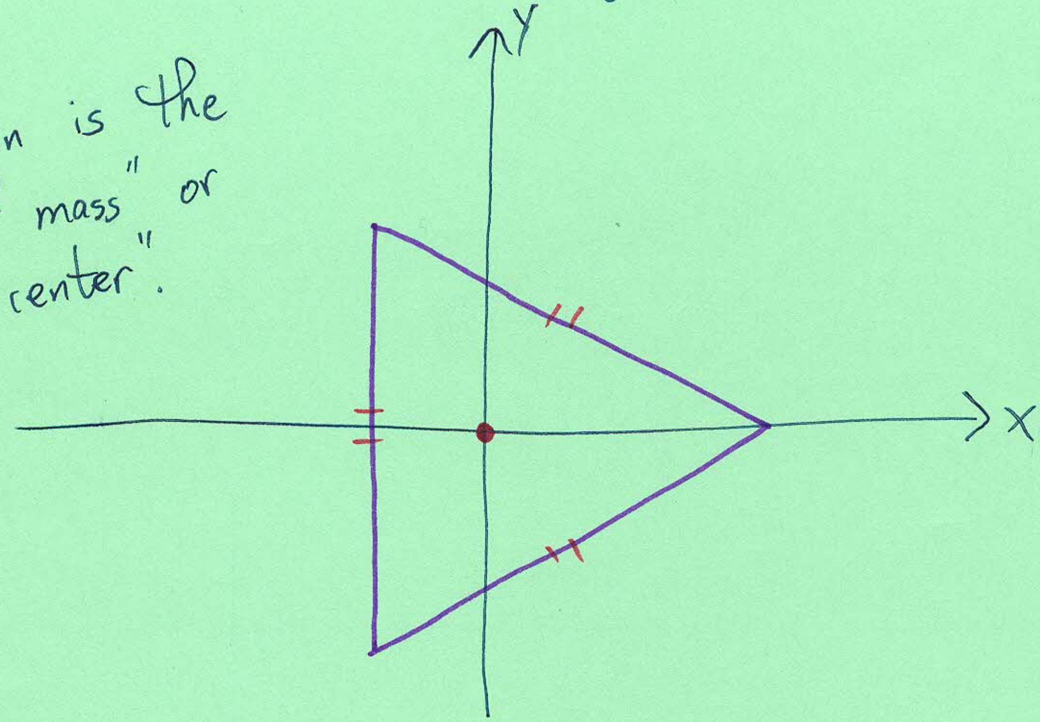
D_n Recall that D_n consisted of rotations by $0, \frac{2\pi}{n}, \frac{4\pi}{n}, \ldots, \frac{2n\pi}{n} = 2\pi$; a certain reflection, and products of these.

Abstractly, we can pick any one reflection S_ψ and generate D_n with $R_{\frac{2\pi}{n}}$ & S_ψ

However, if we have to associate it to a shape, we have to choose S_φ carefully.

Ex: Find the symmetry group of the triangle:

The origin is the "center of mass" or "rotational center".



Sol: Since the x-axis is a line of symmetry, we can use $S_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as the reflection.

The smallest rotation is

$$R_{\frac{2\pi}{3}} = \begin{pmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix} = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$$

Then $R_{\frac{2\pi}{3}}$ & S_0 will generate D_3 for this triangle. Let's verify that: $rs = sr^{-1}$

$$rs = R_{\frac{2\pi}{3}} S_0 = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \left(= S_{\frac{2\pi}{3}}\right)$$

$$sr^{-1} = S_0 R_{-\frac{2\pi}{3}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} = rs \quad \checkmark$$

So far, we have reflections and rotations, but there is a third (and final) type of transformation which preserves distances and angles: translations.

A <u>translation</u> of the plane is a function

$$\tau : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\tau(x,y) = (x + x_0, y + y_0)$$

Notice that we cannot write τ as a 2×2-matrix since $\tau(0,0) = (a,b)$, but $A\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ for any matrix A. To get a matrix for translations, we need to be a little tricky, we need to see the plane inside of 3-space $\mathbb{R}^3 = \{(x,y,z) \mid x,y,z \text{ real \#'s}\}$

You might think we want to use the xy-plane, but $(0,0,0)$ is in the xy-plane, and a matrix will always fix the zero vector. Instead, we think of it as the $z=1$ plane, i.e., the set of points with z-component equal to 1.

So, we're using vectors $\vec{v} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$.

The translation by (x_0, y_0) is thus given by

$$\tau_{(x_0,y_0)} = \begin{pmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\tau_{(x_0,y_0)} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x+0+x_0 \\ 0+y+y_0 \\ 0+0+1 \end{pmatrix} = \begin{pmatrix} x+x_0 \\ y+y_0 \\ 1 \end{pmatrix}$$

In general, matrices which preserve the plane $z=1$ have $(0\ 0\ 1)$ as the bottom row, i.e., matrices of the form

$$\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix}$$

Let's assume this matrix is an isometry, i.e., a rotation, reflection, translation, or combination thereof.

We understand what this matrix does by breaking it into pieces:

$$\left(\begin{array}{cc|c} a \; ① & b & ② \; x_0 \\ c & d & y_0 \\ \hline 0 & 0 & 1 \end{array}\right) = A$$

$$③$$

Part ③ is necessary to stay in the plane $z=1$.

Part ② is a translation by (x_0, y_0).

Part ① is a reflection, rotation, or a product of them,

i.e., $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = R_\theta, S_\psi$

The matrix A performs rotation/reflection <u>before</u> the translation.

$$A\vec{v} = \begin{pmatrix} a & b & x_0 \\ c & d & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} ax+by+x_0 \\ cx+dy+y_0 \\ 1 \end{pmatrix} = \begin{pmatrix} B\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ 1 \end{pmatrix}$$

where $B = R_\theta, S_\psi$